5. Derived

5.1 **Derived definition**

The concept of **Derivative** is at the core of Calculus and modern mathematics. The definition of the derivative can be approached in two different ways. One is geometrical (as a slope of a curve) and the other one is physical (as a rate of change). Historically there was (and maybe still is) a fight between mathematicians which of the two illustrates the concept of the derivative best and which one is more useful. We will not dwell on this and will introduce both concepts. Our emphasis will be on the use of the derivative as a tool.

The Physical Concept of the Derivative
This approach was used by Sir Isaac Newton in the development of his Classical Mechanics. The main idea is the concept of velocity and speed. Indeed, assume you are traveling from point A to point B, what is the average velocity during the trip? It is given by

$$\text{Average velocity} = \frac{\text{distance from A to B}}{\text{time to get from A to B}}$$

If we now assume that A and B are very close to each other, we get close to what is called the **instantaneous velocity**. Of course, if A and B are close to each other, then the time it takes to travel from A to B will also be small. Indeed, assume that at time *t*=*a*, we are at A. If the time elapsed to get to B is Δt, then we will be at B at time $t = a + \Delta t$. If Δs is the distance from A to B, then the average velocity is

$$\text{Average velocity} = \frac{\Delta s}{\Delta t}.$$

The instantaneous velocity (at A) will be found when Δt get smaller and smaller. Here we naturally run into the concept of limit. Indeed, we have

$$\text{Instantaneous Velocity (at A)} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t}.$$

If *f*(*t*) describes the position at time *t*, then $\Delta s = f(a + \Delta t) - f(a)$. In this case, we have

$$\text{Instantaneous Velocity (at A)} = \lim_{\Delta t \to 0} \frac{f(a+\Delta t) - f(a)}{\Delta t}.$$

Also, Just as algebraic geometry is the study of locally modeled on commutative rings, derived algebraic geometry is the study of spaces locally modeled on " derived commutative rings" which means either simplicial commutative rings or equivalently when working over a base field of characteristic zero, non-positively? Graded? Commutative dg-rings. The fundamentals of the subject have been developed by Bertrand Toen and Gabriel Vezzosi, and by Jacob Lurie.

In Gronthendiecks functor of points, approach to the theory of schemes, a scheme over a field k is viewed as a functor from the category CAlg k of commutative k -algebras to the category Set of sets, satisfying a descent condition. Motivated by moduli problems, people have enlarged the target category to the category Grpd of groupoids, arriving at the notion of a stack. Carlos Simpson extended this further by replacing Grpd by the category SSet of simplicial sets, arriving at the notion of higher stacks.

Derived algebraic geometry may be viewed as a further step in this progression, replacing the *source* category CAlg k by SAlg k , the category of simplicial commutative k-algebras. This is motivated by classical concerns involving intersection theory and deformation theory,

Sometimes the term derived algebraic geometry is also used for the related subject of spectral algebraic geometry?, where E-infinity rings are used instead of simplicial rings Derived algebraic geometry may also refer to the study of derived categories of coherent sheaves on varieties as studied in non-commutative algerbraic geometry where one replaces a scheme x by its triangulated category of perfect complexes.

Motivation

There are several motivations for the study of derived algebraic geometry.

1. The hidden smoothness principle of Maxim Kontsevich, which conjectures that in classical algebraic geometry, the non-smoothness? of certain moduli spaces is a consequence of the fact that they are in fact truncations of derived moduli stacks (which are smooth).

2. Universal elliptic cohomology (topological modular forms).
3. Intersection theory?: a geometric interpretation of the Serre intersection formula for non-flat? intersections.
4. Deformation theory?: a geometric interpretation of the cotangent complex. (In derived algebraic geometry, the cotangent complex L X of X *is* its cotangent space.

5.2 **Derived functions**

In mathematics, a **function** is a relation between a set of inputs and a set of permissible outputs with the property that each input is related to exactly one output. **An example** is the function that relates each real number x to its square x^2. The output of a function f corresponding to an input x is denoted by $f(x)$ (read "f of x"). In this example, if the input is −3, then the output is 9, and we may write $f(-3) = 9$. The input variable(s) are sometimes referred to as the argument(s) of the function.

Functions of various kinds are "the central objects of investigation" in most fields of modern mathematics. There are many ways to describe or represent a function. Some functions may be defined by a formula or algorithm that tells how to compute the output for a given input. Others are given by a picture, called the graph of the function. In science, functions are sometimes defined by a table that gives the outputs for selected inputs. A function could be described implicitly, for example as the inverse to another function or as a solution of a differential equation.

The input and output of a function can be expressed as an ordered pair, ordered so that the first element is the input (or tuple of inputs, if the function takes more than one input), and the second is the output. In the example above, $f(x) = x^2$, we have the ordered pair (−3, 9). If both input and output are real numbers, this ordered pair can be viewed as the Cartesian coordinates of a point on the graph of the function. But no picture can exactly define every point in an infinite set.

In modern mathematics, a function is defined by its set of inputs, called the *domain*; a set containing the set of outputs, and possibly additional elements, as members, called its *codomain*; and the set of all input-output pairs, called its *graph*. (Sometimes the codomain is called the function's "range", but **warning**: the word "range" is sometimes used to mean, instead, specifically the set of outputs. An unambiguous word for the latter meaning is the function's "image". To avoid ambiguity, the words "codomain" and "image" are the preferred language for their concepts.)

For example, we could define a function using the rule $f(x) = x^2$ by saying that the domain and codomain are the real numbers, and that the graph consists of all pairs of real numbers (x, x^2). Collections of functions with the same domain and the same codomain are called function spaces, the properties of which are studied in such mathematical disciplines as real analysis, complex analysis, and functional analysis.

In Derived Functions, the derivative of a function of a real variable measures the sensitivity to change of a quantity (a function or dependent variable) which is determined by another quantity (the independent variable). It is a fundamental tool of calculus. For example, the derivative of the position of a moving object with respect to time is the object's velocity: this measures how quickly the position of the object changes when time is advanced. The derivative measures the *instantaneous* rate of change of the function, as distinct from its *average* rate of change, and is defined as the limit of the average rate of change in the function as the length of the interval on which the average is computed tends to zero.

The derivative of a function at a chosen input value describes the best linear approximation of the function near that input value. In fact, the derivative at a point of a function of a single variable is the slope of the tangent line to the graph of the function at that point.

The notion of derivative may be generalized to functions of several real variables. The generalized derivative is a linear map called the differential. Its matrix representation is the Jacobian matrix, which reduces to the gradient vector in the case of real-valued function of several variables.

The process of finding a derivative is called **differentiation**. The reverse process is called *anti-differentiation*. The fundamental theorem of calculus states that anti-differentiation is the same as integration. Differentiation and integration constitute the two fundamental operations in single-variable calculus.

The concept of a derivative can be extended to many other settings. The common thread is that the derivative of a function at a point serves as a linear approximation of the function at that point.

- An important generalization of the derivative concerns complex functions of complex variables, such as functions from (a domain in) the complex numbers **C** to **C**. The notion of the derivative of such a function is obtained by replacing real variables with complex variables in the definition. If **C** is identified with $\mathbf{R}^2$ by writing a complex number z as $x + i\, y$, then a differentiable function from **C** to **C** is certainly differentiable as a function from $\mathbf{R}^2$ to $\mathbf{R}^2$ (in the sense that its partial derivatives all exist), but the converse is not true in general: the complex derivative only exists if the real derivative is *complex linear* and this imposes relations between the partial derivatives called the Cauchy Riemann equations – see holomorphic functions.

- Another generalization concerns functions between differentiable or smooth manifolds. Intuitively speaking such a manifold M is a space that can be approximated near each point x by a vector space called its tangent space: the prototypical example is a smooth surface in $\mathbf{R}^3$. The derivative (or differential) of a (differentiable) map $f: M \to N$ between manifolds, at a point x in M, is then a linear map from the tangent space of M at x to the tangent space of N at $f(x)$. The derivative function becomes a map between the tangent bundles of M and N. This definition is fundamental in differential geometry and has many uses – see pushforward (differential) and pullback (differential geometry).
- Differentiation can also be defined for maps between infinite dimensional vector spaces such as Banach spaces and Fréchet spaces. There is a generalization both of the directional derivative, called the Gâteaux derivative, and of the differential, called the Fréchet derivative.
- One deficiency of the classical derivative is that not very many functions are differentiable. Nevertheless, there is a way of extending the notion of the derivative so that all continuous functions and many other functions can be differentiated using a concept known as the weak derivative. The idea is to embed the continuous functions in a larger space called the space of distributions and only require that a function is differentiable "on average".
- The properties of the derivative have inspired the introduction and study of many similar objects in algebra and topology — see, for example, differential algebra.
- The discrete equivalent of differentiation is finite differences. The study of differential calculus is unified with the calculus of finite differences in time scale calculus.